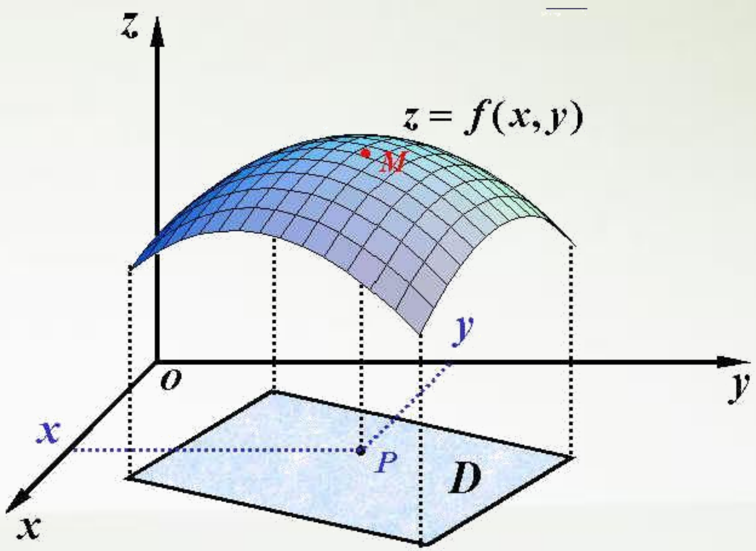
高等数学



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第二节

不定积分的换元法

一、第一类换元法

问题 $\int \cos 2x dx \overset{?}{=} \sin 2x + C,$

解决方法 利用复合函数，设置中间变量.

过程 令 $t = 2x \Rightarrow dx = \frac{1}{2} dt,$

$$\int \cos 2x dx = \frac{1}{2} \int \cos t dt = \frac{1}{2} \sin t + C = \frac{1}{2} \sin 2x + C.$$

在一般情况下：

设 $F'(u)=f(u)$，则 $\int f(u)du=F(u)+C.$

如果 $u=\varphi(x)$（可微）

$\because \quad dF[\varphi(x)]=f[\varphi(x)]\varphi'(x)dx$

$\therefore \quad \int f[\varphi(x)]\varphi'(x)dx=F[\varphi(x)]+C$

$=[\int f(u)du]_{u=\varphi(x)}$ 由此可得换元法定理

定理1 设 $f(u)$ 具有原函数，$u=\varphi(x)$ 可导，

则有换元公式

$$\int f[\varphi(x)]\varphi'(x)dx=[\int f(u)du]_{u=\varphi(x)}$$

第一类换元公式（凑微分法）

说明 使用此公式的关键在于将

$$\int g(x)dx \text{ 化为} \int f[\varphi(x)]\varphi'(x)dx.$$

观察重点不同，所得结论不同.

例1　求$\int \sin 2x dx$.

解（一）$\int \sin 2x dx = \frac{1}{2}\int \sin 2x d(2x)$

$= -\frac{1}{2}\cos 2x + C$;

解（二）$\int \sin 2x dx = 2\int \sin x \cos x dx$

$= 2\int \sin x d(\sin x) = (\sin x)^2 + C$;

解（三）$\int \sin 2x dx = 2\int \sin x \cos x dx$

$= -2\int \cos x d(\cos x) = -(\cos x)^2 + C$.

例2 求$\int \frac{1}{3+2x}dx.$

解 $\frac{1}{3+2x} = \frac{1}{2} \cdot \frac{1}{3+2x} \cdot (3+2x)',$

$$\int \frac{1}{3+2x}dx = \frac{1}{2}\int \frac{1}{3+2x} \cdot (3+2x)'dx$$

$$= \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln u + C = \frac{1}{2}\ln(3+2x) + C.$$

一般地 $\int f(ax+b)dx = \frac{1}{a}\left[\int f(u)du\right]_{u=ax+b}$

例3 求 $\int \frac{1}{x(1+2\ln x)}dx.$

解 $\int \frac{1}{x(1+2\ln x)}dx = \int \frac{1}{1+2\ln x}d(\ln x)$

$= \frac{1}{2}\int \frac{1}{1+2\ln x}d(1+2\ln x)$

$u = 1+2\ln x$

$= \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln u + C = \frac{1}{2}\ln(1+2\ln x) + C.$

例4 求$\int \frac{x}{(1+x)^3}dx.$

解 $\int \frac{x}{(1+x)^3}dx = \int \frac{x+1-1}{(1+x)^3}dx$

$$= \int \left[\frac{1}{(1+x)^2} - \frac{1}{(1+x)^3}\right]d(1+x)$$

$$= -\frac{1}{1+x} + C_1 + \frac{1}{2(1+x)^2} + C_2$$

$$= -\frac{1}{1+x} + \frac{1}{2(1+x)^2} + C.$$

例5 求 $\int \frac{1}{a^2+x^2}dx.$

解 $\int \frac{1}{a^2+x^2}dx = \frac{1}{a^2}\int \frac{1}{1+\frac{x^2}{a^2}}dx$

$$= \frac{1}{a}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}d\left(\frac{x}{a}\right) = \frac{1}{a}\arctan\frac{x}{a}+C.$$

例6 求$\int \frac{1}{x^2-8x+25}dx.$

解 $\int \frac{1}{x^2-8x+25}dx = \int \frac{1}{(x-4)^2+9}dx$

$$= \frac{1}{3^2}\int \frac{1}{\left(\frac{x-4}{3}\right)^2+1}dx = \frac{1}{3}\int \frac{1}{\left(\frac{x-4}{3}\right)^2+1}d\left(\frac{x-4}{3}\right)$$

$$= \frac{1}{3}\arctan\frac{x-4}{3}+C.$$

例7 求$\int \frac{1}{1+e^x}dx$.

解 $\int \frac{1}{1+e^x}dx = \int \frac{1+e^x-e^x}{1+e^x}dx$

$$= \int\left(1-\frac{e^x}{1+e^x}\right)dx = \int dx - \int \frac{e^x}{1+e^x}dx$$

$$= \int dx - \int \frac{1}{1+e^x}d(1+e^x)$$

$$= x - \ln(1+e^x) + C.$$

例8 求$\int (1-\frac{1}{x^2})e^{x+\frac{1}{x}}dx.$

解 $\because \left(x+\frac{1}{x}\right)' = 1-\frac{1}{x^2},$

$\therefore \int (1-\frac{1}{x^2})e^{x+\frac{1}{x}}dx$

$= \int e^{x+\frac{1}{x}}d(x+\frac{1}{x}) = e^{x+\frac{1}{x}} + C.$

例9 求 $\int \frac{1}{\sqrt{2x+3}+\sqrt{2x-1}}dx.$

$$原式=\int \frac{\sqrt{2x+3}-\sqrt{2x-1}}{\left(\sqrt{2x+3}+\sqrt{2x-1}\right)\left(\sqrt{2x+3}-\sqrt{2x-1}\right)}dx$$

$$=\frac{1}{4}\int \sqrt{2x+3}dx-\frac{1}{4}\int \sqrt{2x-1}dx$$

$$=\frac{1}{8}\int \sqrt{2x+3}d(2x+3)-\frac{1}{8}\int \sqrt{2x-1}d(2x-1)$$

$$=\frac{1}{12}\left(\sqrt{2x+3}\right)^3-\frac{1}{12}\left(\sqrt{2x-1}\right)^3+C.$$

例10 求$\int \frac{1}{1+\cos x}dx.$

解 $\int \frac{1}{1+\cos x}dx = \int \frac{1-\cos x}{(1+\cos x)(1-\cos x)}dx$

$$= \int \frac{1-\cos x}{1-\cos^2 x}dx = \int \frac{1-\cos x}{\sin^2 x}dx$$

$$= \int \frac{1}{\sin^2 x}dx - \int \frac{1}{\sin^2 x}d(\sin x)$$

$$= -\cot x + \frac{1}{\sin x} + C.$$

例11 求 $\int \sin^2 x \cdot \cos^5 x dx$.

解 $\int \sin^2 x \cdot \cos^5 x dx = \int \sin^2 x \cdot \cos^4 x d(\sin x)$

$= \int \sin^2 x \cdot (1-\sin^2 x)^2 d(\sin x)$

$= \int (\sin^2 x - 2\sin^4 x + \sin^6 x) d(\sin x)$

$= \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C.$

说明 当被积函数是三角函数相乘时，拆开奇次项去凑微分.

例12 求$\int \cos 3x \cos 2x dx.$

解 $\cos A \cos B = \frac{1}{2}[\cos(A-B)+\cos(A+B)],$

$$\cos 3x \cos 2x = \frac{1}{2}(\cos x + \cos 5x),$$

$$\int \cos 3x \cos 2x dx = \frac{1}{2}\int (\cos x + \cos 5x) dx$$

$$= \frac{1}{2}\sin x + \frac{1}{10}\sin 5x + C.$$

例13 求$\int \csc x dx$.

解（一）$\int \csc x dx = \int \frac{1}{\sin x} dx = \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} dx$

$$= \int \frac{1}{\tan\frac{x}{2}\left(\cos\frac{x}{2}\right)^2} d\left(\frac{x}{2}\right) = \int \frac{1}{\tan\frac{x}{2}} d\left(\tan\frac{x}{2}\right)$$

$$= \ln\tan\frac{x}{2} + C = \ln(\csc x - \cot x) + C.$$

（使用了三角函数恒等变形）

解（二）$\int \csc x dx = \int \frac{1}{\sin x} dx = \int \frac{\sin x}{\sin^2 x} dx$

$$= -\int \frac{1}{1-\cos^2 x} d(\cos x) \qquad u = \cos x$$

$$= -\int \frac{1}{1-u^2} du = -\frac{1}{2}\int \left(\frac{1}{1-u} + \frac{1}{1+u} \right) du$$

$$= \frac{1}{2} \ln \frac{1-u}{1+u} + C = \frac{1}{2} \ln \frac{1-\cos x}{1+\cos x} + C.$$

类似地可推出$\int \sec x dx = \ln(\sec x + \tan x) + C.$

例14 设 $f'(\sin^2 x)=\cos^2 x$，求 $f(x)$.

解 令 $u=\sin^2 x \Rightarrow \cos^2 x=1-u$,

$$f'(u)=1-u,$$

$$f(u)=\int(1-u)du=u-\frac{1}{2}u^2+C,$$

$$f(x)=x-\frac{1}{2}x^2+C.$$

例15 求$\int \frac{1}{\sqrt{4-x^2}\arcsin\frac{x}{2}}dx.$

解 $\int \frac{1}{\sqrt{4-x^2}\arcsin\frac{x}{2}}dx = \int \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}\arcsin\frac{x}{2}}d\frac{x}{2}$

$= \int \frac{1}{\arcsin\frac{x}{2}}d(\arcsin\frac{x}{2}) = \ln\arcsin\frac{x}{2} + C.$

二、第二类换元法

问题 $\int x^5\sqrt{1-x^2}dx = ?$

解决方法 改变中间变量的设置方法.

过程 令 $x = \sin t \Rightarrow dx = \cos t dt,$

$$\int x^5\sqrt{1-x^2}dx = \int (\sin t)^5\sqrt{1-\sin^2 t}\cos t dt$$

$$= \int \sin^5 t \cos^2 t dt = \cdots\cdots$$

（应用"凑微分"即可求出结果）

定理2　设$x=\psi(t)$是单调的、可导的函数，并且$\psi'(t)\neq 0$，又设$f[\psi(t)]\psi'(t)$具有原函数，则有换元公式$\int f(x)dx=\left[\int f[\psi(t)]\psi'(t)dt\right]_{t=\overline{\psi}(x)}$

其中$\overline{\psi}(x)$是$x=\psi(t)$的反函数.

证　设$\Phi(t)$为$f[\psi(t)]\psi'(t)$的原函数,

令$F(x)=\Phi[\overline{\psi}(x)]$

则 $F'(x)=\dfrac{d\Phi}{dt}\cdot\dfrac{dt}{dx}=f[\psi(t)]\psi'(t)\cdot\dfrac{1}{\psi'(t)}$,

$$= f[\psi(t)] = f(x).$$

说明$F(x)$为$f(x)$的原函数,

$$\therefore \int f(x)dx = F(x) + C = \Phi[\overline{\psi}(x)] + C,$$

$$\int f(x)dx = \left[\int f[\psi(t)]\psi'(t)dt\right]_{t=\overline{\psi}(x)}$$

第二类积分换元公式

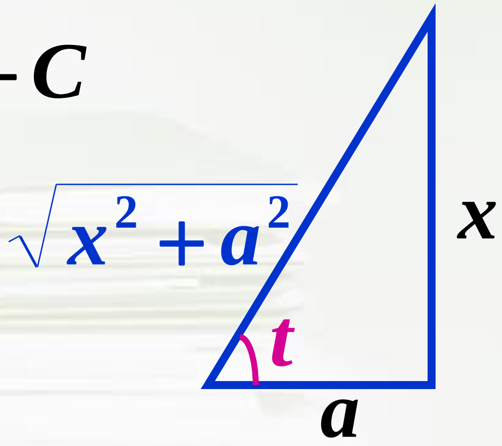
例16　求 $\int \frac{1}{\sqrt{x^2+a^2}}dx \quad (a>0)$.

解　令 $x = a\tan t \Rightarrow dx = a\sec^2 t dt \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \int \frac{1}{a\sec t}\cdot a\sec^2 t dt$$

$$= \int \sec t dt = \ln(\sec t + \tan t) + C$$

$$= \ln\left(\frac{x}{a} + \frac{\sqrt{x^2+a^2}}{a}\right) + C.$$



例17　求 $\int x^3\sqrt{4-x^2}dx$.

解　令 $x=2\sin t$　　$dx=2\cos t dt$　　$t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

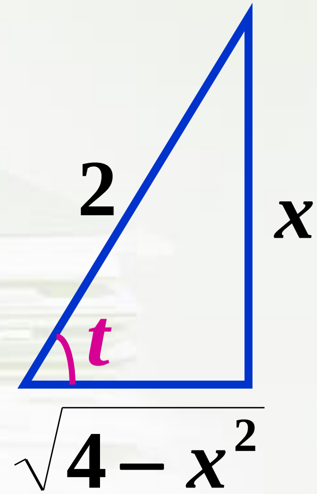
$$\int x^3\sqrt{4-x^2}dx=\int(2\sin t)^3\sqrt{4-4\sin^2 t}\cdot 2\cos t dt$$

$$=32\int\sin^3 t\cos^2 t dt=32\int\sin t(1-\cos^2 t)\cos^2 t dt$$

$$=-32\int(\cos^2 t-\cos^4 t)d\cos t$$

$$=-32\left(\frac{1}{3}\cos^3 t-\frac{1}{5}\cos^5 t\right)+C$$

$$=-\frac{4}{3}\left(\sqrt{4-x^2}\right)^3+\frac{1}{5}\left(\sqrt{4-x^2}\right)^5+C.$$



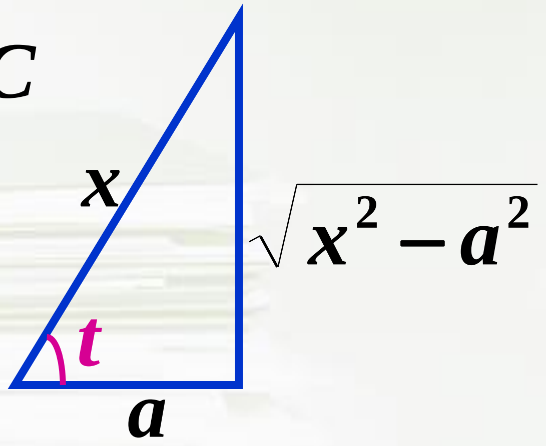
例18 求$\int \frac{1}{\sqrt{x^2-a^2}}dx \quad (a>0).$

解 令$x = a\sec t \quad dx = a\sec t\tan t dt \quad t\in\left(0,\frac{\pi}{2}\right)$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \int \frac{a\sec t\cdot\tan t}{a\tan t}dt$$

$$=\int \sec t dt = \ln(\sec t+\tan t)+C$$

$$=\ln\left(\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right)+C.$$



说明(1) 以上几例所使用的均为三角代换.

三角代换的目的是化掉根式.

一般规律如下：当被积函数中含有

(1) $\sqrt{a^2 - x^2}$ 可令 $x = a\sin t$;

(2) $\sqrt{a^2 + x^2}$ 可令 $x = a\tan t$;

(3) $\sqrt{x^2 - a^2}$ 可令 $x = a\sec t$.

说明(2) 积分中为了化掉根式除采用三角代换外还可用**双曲代换.**

$$\because \cosh^2 t - \sinh^2 t = 1$$

$\therefore\ x = a\sinh t,\quad x = a\cosh t$ 也可以化掉根式

例 $\int \frac{1}{\sqrt{x^2+a^2}}dx$ 中, 令 $x = a\sinh t$ $\quad dx = a\cosh t dt$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \int \frac{a\cosh t}{a\cosh t}dt = \int dt = t + C$$

$$= ar\sinh\frac{x}{a} + C = \ln\left(\frac{x}{a} + \frac{\sqrt{x^2+a^2}}{a}\right) + C.$$

说明(3) 积分中为了化掉根式是否一定采用三角代换（或双曲代换）并不是绝对的，需根据被积函数的情况来定.

例19 求$\int \frac{x^5}{\sqrt{1+x^2}}dx$ **（三角代换很繁琐）**

解 令$t=\sqrt{1+x^2} \Rightarrow x^2=t^2-1$, $xdx=tdt$,

$$\int \frac{x^5}{\sqrt{1+x^2}}dx=\int \frac{\left(t^2-1\right)^2}{t}tdt=\int \left(t^4-2t^2+1\right)dt$$

$$=\frac{1}{5}t^5-\frac{2}{3}t^3+t+C=\frac{1}{15}(8-4x^2+3x^4)\sqrt{1+x^2}+C.$$

例20 求$\int \frac{1}{\sqrt{1+e^x}}dx.$

解 令$t=\sqrt{1+e^x} \Rightarrow e^x = t^2-1,$

$$x=\ln(t^2-1), \quad dx=\frac{2t}{t^2-1}dt,$$

$$\int \frac{1}{\sqrt{1+e^x}}dx=\int \frac{2}{t^2-1}dt=\int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt$$

$$=\ln\left|\frac{t-1}{t+1}\right|+C=2\ln\left(\sqrt{1+e^x}-1\right)-x+C.$$

说明(4) 当分母的阶较高时, 可采用倒代换 $x=\frac{1}{t}$.

例21 求 $\int\frac{1}{x(x^7+2)}dx$

解 令 $x=\frac{1}{t}\Rightarrow dx=-\frac{1}{t^2}dt,$

$$\int\frac{1}{x(x^7+2)}dx=\int\frac{t}{\left(\frac{1}{t}\right)^7+2}\cdot\left(-\frac{1}{t^2}\right)dt=-\int\frac{t^6}{1+2t^7}dt$$

$$=-\frac{1}{14}\ln|1+2t^7|+C=-\frac{1}{14}\ln|2+x^7|+\frac{1}{2}\ln|x|+C.$$

例22 求$\int \frac{1}{x^4\sqrt{x^2+1}}dx.$ （分母的阶较高）

解 令 $x=\frac{1}{t} \Rightarrow dx=-\frac{1}{t^2}dt,$

$$\int \frac{1}{x^4\sqrt{x^2+1}}dx=\int \frac{1}{\left(\frac{1}{t}\right)^4\sqrt{\left(\frac{1}{t}\right)^2+1}}\left(-\frac{1}{t^2}\right)dx$$

$$=-\int \frac{t^3}{\sqrt{1+t^2}}dt=-\frac{1}{2}\int \frac{t^2}{\sqrt{1+t^2}}dt^2 \qquad u=t^2$$

$$=-\frac{1}{2}\int\frac{u}{\sqrt{1+u}}du=\frac{1}{2}\int\frac{1-1-u}{\sqrt{1+u}}du$$

$$=\frac{1}{2}\int\left(\frac{1}{\sqrt{1+u}}-\sqrt{1+u}\right)d(1+u)$$

$$=-\frac{1}{3}\left(\sqrt{1+u}\right)^3+\sqrt{1+u}+C$$

$$=-\frac{1}{3}\left(\frac{\sqrt{1+x^2}}{x}\right)^3+\frac{\sqrt{1+x^2}}{x}+C.$$

说明(5) 当被积函数含有两种或两种以上的根式 $\sqrt[k]{x},\cdots,\sqrt[l]{x}$ 时，可采用令 $x=t^n$（其中 n 为各根指数的**最小公倍数**）

例23 求 $\int \frac{1}{\sqrt{x}(1+\sqrt[3]{x})}dx.$

解 令 $x=t^6 \Rightarrow dx=6t^5dt,$

$$\int \frac{1}{\sqrt{x}(1+\sqrt[3]{x})}dx=\int \frac{6t^5}{t^3(1+t^2)}dt=\int \frac{6t^2}{1+t^2}dt$$

$$= 6\int \frac{t^2+1-1}{1+t^2}dt$$

$$= 6\int \left(1-\frac{1}{1+t^2}\right)dt$$

$$= 6[t - \arctan t] + C$$

$$= 6[\sqrt[6]{x} - \arctan \sqrt[6]{x}] + C.$$

基本积分表⑵

(16) $\int \tan x dx = -\ln \cos x + C;$

(17) $\int \cot x dx = \ln \sin x + C;$

(18) $\int \sec x dx = \ln(\sec x + \tan x) + C;$

(19) $\int \csc x dx = \ln(\csc x - \cot x) + C;$

(20) $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C;$

(21) $$\int \frac{1}{x^2-a^2}dx=\frac{1}{2a}\ln\frac{x-a}{x+a}+C;$$

(22) $$\int \frac{1}{a^2-x^2}dx=\frac{1}{2a}\ln\frac{a+x}{a-x}+C;$$

(23) $$\int \frac{1}{\sqrt{a^2-x^2}}dx=\arcsin\frac{x}{a}+C;$$

(24) $$\int \frac{1}{\sqrt{x^2\pm a^2}}dx=\ln(x+\sqrt{x^2\pm a^2})+C.$$

三、小结

两类积分换元法：

- **（一）凑微分**
- **（二）三角代换、倒代换、根式代换**

基本积分表(2)